Limit analysis for civil engineering structures

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Abstract

The analysis of ultimate limit state (ULS) of a structure requires a stability study until failure. This mechanical behaviour is complex to compute with standard tools. Cracking, damage, elastic-plastic law, etc., are phenomena which often lead to numerical problem of convergence and interpretation of results. It is therefore often advised to use codes instead (Eurocodes, AASHTO, etc.), but this solution comes at the expense of accurate analysis of the physical behaviour of failure. An alternate solution is limit analysis, which combines two parallel and complementary methods. Used on a finite element mesh for rigid-plastic calculations, these two methods lead to a full determination of the physical failure: mechanism, stresses distribution and safety factor. Strains presents a software program using limit analysis for steel beam connections nodes, taking into account such phenomena as contact, separation, friction, welding, plasticity and pre-stressed bolts.

Keywords: limit analysis; yield design; civil engineering; failure; safety; steel profiles.

1 Introduction

In order to check the stability of a structure at failure, engineers typically use the concept of ultimate limit state (ULS) as defined in various codes (Eurocodes, AASHTO, SNIP, etc.). This state is often studied under an elastic hypothesis, and doesn't take into account all nonlinear phenomena linked to failure: plasticity, cracking, damage, etc.

When possible, a simplified approach used by engineers is to perform elastic studies mainly by using simple software programs or doing manual calculations. This seldom takes into account nonlinear aspects. Instead, safety is generally built in the computations by increasing loads and by curbing the limit strength of the materials, following rules provided in the codes.

The drawback of this method is that it does not take into account the physical behaviour of the structure. As said previously, failure is typically nonlinear and an elastic analysis, even when safety factors are included, does not account for the real physical behaviour (displacements and stresses).

Hence, if necessary, engineers have to perform elasto-plastic analysis. Not only can this take time in order to create the full 3D model, especially in the field of metallic beam connections, but this often leads to numerical problem of convergence, and complex interpretation of results.

Another way to assess the failure state is by using the limit analysis theory.

2 From limit analysis

By definition, limit analysis aims at studying a structure at its limits, meaning at failure, by assuming all materials have reached (and withstood) their limit strength criterion. The elastic behaviour is therefore not included in the analysis, therefore no elasto-plastic iterations need to be performed. But the underlying assumption is that the materials allow high ductility deformations.

The plasticity is defined thanks to a criterion which limits the stresses. It's usually defined by a function f over an admissible stress value domain G (1).

$$\sigma \in G \Leftrightarrow f(\sigma) \leq 0 \qquad (1)$$

As material criterion (or law), here is a list of few examples:

- $N_c \leq N \leq N_t$ for 1d beam under normal force
- $M_1 \leq M \leq M_2$ for 1d beam under moment
- Von Mises for 3d steel model
- Drucker-Prager for 3d soil model

More details about limit analysis can be found in [1] and [2].

This theory, developed in the 80s, is currently used in very specific fields. In soil mechanic when the engineer makes a guess on a failure mechanism by blocs separated by logarithmic spirals, or in reinforced concrete when the engineer needs to find a strut-and-tie pattern as in Figure 1.

This theory is not widely used in civil engineering because of its main drawback: the engineer (or the software program) has to make a guess on what the solution is likely to be: either a failure mechanism (by blocs for instance), or a stress distribution (as in strut-and-tie method).

Most of the time engineers don't know in advance the solution, so they need to try different solutions until finding the optimal one. As this can be very tricky, especially in case of complex structures, this theory has limited use (2D analysis of examples above), and will need careful review by senior engineers.

In short, limit analysis is the research of an optimal solution either in stress distribution or in failure displacements. That is why it's typically split in two approaches: a static one and a kinematic one.

2.1 Static approach

In a static approach and for a given external load *F*, one wants to find the safety factor λ which leads to failure. To do so, a first stress distribution σ^1, in equilibrium with F, is guessed. Then, the maximal multiplier α^1, such that $\alpha^1\sigma^1$ withstands the given material criterion $f(\alpha^1\sigma^1) \leq 0$, is found.

As only one stress distribution has been analysed, one does not know if it is the optimal distribution. Hence $\alpha^1 \leq \lambda$, and, by implementing this method with another distribution σ^2, still in equilibrium with *F*, one gets α^2 which can be higher or lower than α^1.

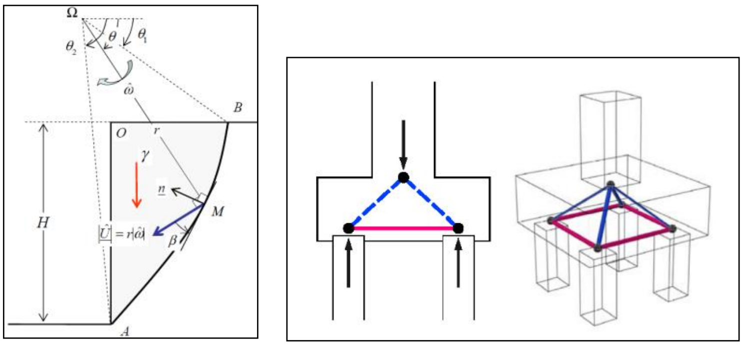

Figure 1. Limit analysis examples. Left: collapse of the rigid block OAB under its weight as in [1]. Right: strut-and-tie pattern of a reinforced concrete piece as in [3].

By trying all admissible distributions in equilibrium with *F*, the static approach leads to the calculation of the maximum of all multipliers α^i. And as all mathematical possible distributions cannot be tried, $\alpha^{opt} = \max(\alpha^i)$ is computed as a lower bound of λ.

The output of this approach is α^{opt}, lower bound of the load multiplier, and the corresponding stress distribution. The drawback is the lack of information regarding the displacement field.

2.2 Kinematic approach

The kinematic approach is based on the research of an optimal displacements field which minimizes the deformation energy. For a given external load *F*, the Virtual Work Principle (VWP) over the structure gives (2):

$$\forall \hat{U} \quad F.\hat{U} = \iiint_V \sigma : \varepsilon(\hat{U}) dV \qquad (2)$$

As, by hypothesis, all material have reached their limit strength criterion (3):

$$\sigma : \varepsilon(\hat{U}) \leq \max[\sigma : \varepsilon(\hat{U}), \forall \sigma \in G] \stackrel{\text{def}}{=} \pi(\hat{U}) \qquad (3)$$

where π is the support function of the criterion.

Equations (2) and (3) combined leads to this inequality (4):

$$\forall \hat{U} \quad F.\hat{U} \leq \iiint_V \pi(\hat{U}) dV \qquad (4)$$

If it's verified for all virtual displacements $\hat{U}$, the structure withstands the load *F*. On the contrary, if at least one displacement $\widehat{U_0}$ that does not verify this inequality is found, that means the structure will collapse under *F*, with $\widehat{U_0}$ as failure mechanism.

Let's define β as follow (5):

$$\beta(\hat{U}) = \frac{\iiint_V \pi(\hat{U}) dV}{F.\hat{U}} \Rightarrow \forall \hat{U} \ \beta(\hat{U}) \geq 1 \qquad (5)$$

So, $\beta^{opt} = \min \beta(\hat{U}) \ \forall \hat{U}$ is computed. Two possibilities appear:

- $\beta^{opt} \leq 1 \Rightarrow$ collapse of the structure under *F*
- $\beta^{opt} > 1 \Rightarrow$ the structure withstands *F*

But, (5) can also be written (6):

$$\beta^{opt} = \min \frac{\iiint_V \pi(\hat{U}) dV}{F.\hat{U}} \Rightarrow \min \frac{\iiint_V \pi(\hat{U}) dV}{\beta^{opt} F.\hat{U}} = 1 \qquad (6)$$

Hence, β^{opt} can be seen in (6) as the load multiplier to the failure. As in the static approach, all possible $\hat{U}$ cannot be tried, and β^{opt} is calculated as an upper bound of the safety factor $\lambda \leq \beta^{opt}$.

The output of the kinematic approach is an upper bound of the safety factor and the corresponding failure mechanism. The drawback is the lack of information regarding the stress distribution.

2.3 Optimal safety factor

The combination of these two approaches, provides the bounds of the safety factor λ (7):

$$\alpha^{opt} \leq \lambda \leq \beta^{opt} \qquad (7)$$

An optimal analysis would give the same value for the upper and lower bounds. But, as all possible solutions cannot be tried, the equality of the bounds is seldom found.

Hence, the precision on the safety factor depends on the values of both bounds: the closer they are, the more precise the safety factor is.

Therefore, the limit analysis is the perfect tool to assess the real physical behaviour of ULS.

3 To a steel nodes software

Few software programs use limit analysis. The classic field is in soil mechanics, to check the stability of the ground. But, not only do they usually stay in 2D, but they also focus on the kinematic approach. This is due to the numerical method which often uses strict displacement field hypothesis as shown in Figure 2.

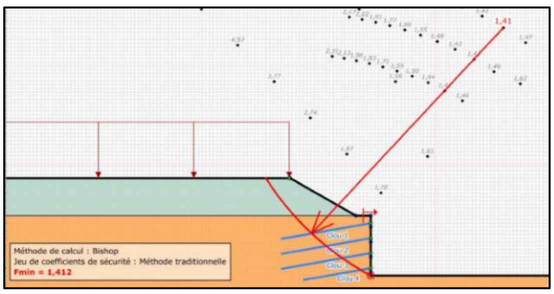

Figure 2. Caption of Talren software as in [4].

This issue is at stake in several disciplines of civil engineering. Strains is therefore seeking to expand the scope of use of limit analysis to a new field: steel beam connection nodes.

3.1 Objectives

This software solution aims at analysing the stability of different kinds of steel beam connections nodes (bridges trusses or buildings frameworks for instance) in a full 3D model.

Thanks to a meshed geometry, with given boundary conditions (in displacements and forces), the static and kinematic approaches are computed via a very efficient external optimizer tool. As output, the failure behaviour is obtained.

3.2 Phenomena taken into account

In order to be realistic, the physical behaviour of all components of the structure has to be taken into account:

- Rigid-plastic behaviour of steel (Von Mises)
- Welded pieces
- Bolts (pre-stressed or not)
- Friction on contact areas

A major difficulty resides in the friction behaviour, because of the non-associated rule.

3.3 Friction behaviour

In theory, the plastic flow must be normal to the criterion surface (“admissible $\hat{U}$ area”), which leads to the loss of contact in case of friction. However, a volume can slide tangentially to another volume, without losing contact (Figure 3).

As limit analysis is based on normal plastic flow rule, a modified law is used to represent the real behaviour of friction.

3.4 Calculation

To perform an efficient analysis without constraining the admissible guessed fields, solid finite elements are created to assemble all the data necessary to calculate both extrema of the static and kinematic approaches.

The mathematical problem can also be written in terms of conic quadratic optimization, which can be solved very quickly by an external software solution used for this purpose.

The static approach is done first to assess the stresses distribution and to calculate the normal force on all contact areas. Then, according to this first result, the kinematic problem is solved to get the displacements field, and so, the failure mechanism.

3.5 Anisotropic mesh refinement

A key feature of finite elements is the ability to refine the mesh according to a first analysis. This is done automatically, by focusing on the more stressed or deformed areas to be refined, and by letting loose the rigid parts.

Moreover, thanks to an anisotropic mesh, solid elements are distorted on purpose, along one direction of no deformation. Hence it’s possible to have a refined mesh over a beam cross section, but a coarse mesh along the beam axis.

This saves a lot of engineer-time and computer-time.

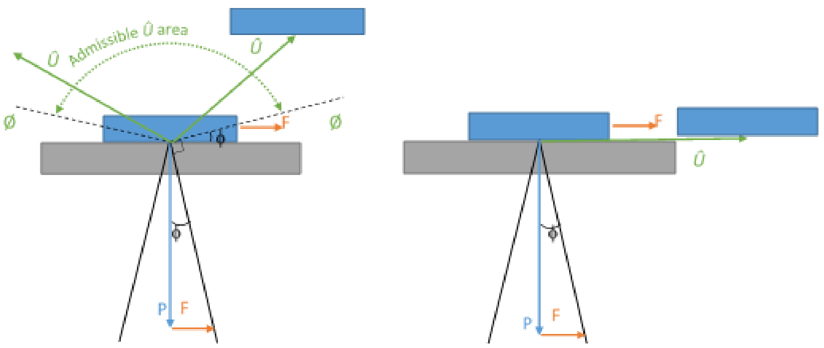

Figure 3. Frictional behaviour: associated law (left), non-associated law (right).P is the normal force, F the friction force and φ the friction angle.

3.6 Output

Both approaches combined provides full knowledge mechanism behaviour with the stress distribution, displacements field and the bounds of the safety factor. An example of the kinematic approach output is shown in Figure 4 (using GMSH viewer for development only).

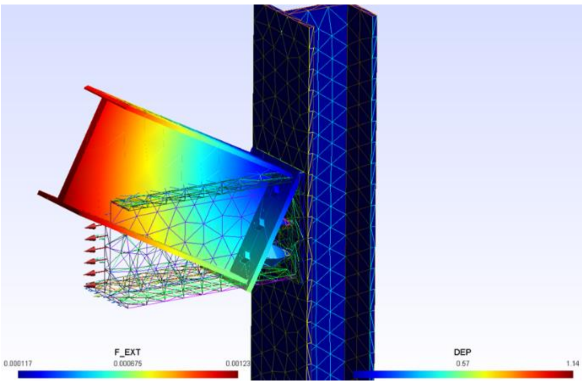

Figure 4. Failure of a bolted beam in traction (the torque is due to a chamfered surface)

4 Conclusions

The combination of limit analysis theory, supported by a new user-friendly CAD, a powerful 3d solid mesher and a very efficient conic quadratic optimizer leads to a modern tool, available in SaaS mode, for civil engineers that allows the checking of steel beam connection nodes.

Strains aims at expanding the scope of use of limit analysis by creating more tools based on the limit analysis theory. For instance, 3d soil stability or 3d reinforced concrete massive pieces.

5 References

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